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LETTER TO THE EDITOR

Some remarks on the commutation relations for angle-angular momentum variables in quantum mechanics

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Abstract. A commutation relation for the observables corresponding to the azimuthal angle about the z axis and the z component of the orbital angular momentum in wave mechanics is presented which will be valid on the entire Hilbert space of states.

It is well known that the relation

$$\mathrm{i}[\hat{P}, \hat{Q}] = \hat{I} \tag{1}$$

for $\hat{P}$, $\hat{Q}$ self-adjoint operators acting on a Hilbert space $\mathscr{H}$ cannot be valid for all the elements of $\mathscr{H}$. It may, however, be valid on a dense subset of $\mathscr{H}$.

For the Schrödinger couple $(-\mathrm{i}\partial_x, x)$ acting on $L^2(-\infty, \infty)$ we have a second relation

$$\mathrm{e}^{-\mathrm{i}\alpha\beta}\, \mathrm{e}^{\mathrm{i}\alpha\hat{P}}\, \mathrm{e}^{\mathrm{i}\beta\hat{P}} = \mathrm{e}^{\mathrm{i}\beta\hat{Q}}\, \mathrm{e}^{\mathrm{i}\alpha\hat{P}} \qquad \forall\alpha, \beta \in \mathbb{R}, \tag{2}$$

which in fact is valid on the entire Hilbert space. Von Neumann has shown that (2) can only be satisfied for operators which have essentially the same spectral properties as the Schrödinger couple. Physically speaking the Schrödinger couple corresponds to the momentum and position operators in wave mechanics (see e.g. Prugovecki 1971).

However, we can represent the z component of the orbital angular momentum by $\hat{L} = -\mathrm{i}\partial_\phi$, and the azimuthal angle about the z axis by $\hat{\phi} = \phi$, so that

$$\mathrm{i}[\hat{L}, \hat{\phi}] = \hat{I} \text{ on } \{\psi \mid \psi, \psi' \text{ abs. cont. } \psi(-\pi) = \psi(\pi) = 0\}, \tag{3}$$

which is dense in $L^2(-\pi, \pi)$ (Kraus 1965).

However, $(\hat{L}, \hat{\phi})$ cannot satisfy the commutation relation in the form (2) since this would contradict Von Neumann's theorem.

Kraus (1965) has suggested that an appropriate relation valid on the entire Hilbert space might be

$$\mathrm{e}^{\mathrm{i}\alpha L}\, \mathrm{e}^{\mathrm{i}\beta\phi} = \mathrm{e}^{\mathrm{i}\beta Y(\phi+\alpha)}\, \mathrm{e}^{\mathrm{i}\alpha L}, \tag{4}$$

where $Y(\phi + \alpha) = \phi + \alpha + 2\pi n$, n being an integer which is chosen so that $\phi + \alpha + 2\pi n$ always lies between $-\pi$ and π.

In the form (4) the Kraus relation is somewhat difficult to interpret, since he represented $\hat{L}$ by $-\mathrm{i}\partial_\phi$, $\hat{\phi}$ by ϕ and has not distinguished between the operator $\hat{\phi}$ and the parameter ϕ. The relation would probably be better written

$$\mathrm{e}^{\mathrm{i}\alpha\hat{L}}\, \mathrm{e}^{\mathrm{i}\beta\hat{\phi}}\, \mathrm{e}^{-\mathrm{i}\alpha\hat{L}}\, \mathrm{e}^{-\mathrm{i}\beta\hat{\phi}}\, \mathscr{S}(\phi) = \mathrm{e}^{\mathrm{i}\beta(Y(\phi+\alpha)-Y(\phi))}\mathscr{S}(\phi). \tag{5}$$

Equation (5) reduces to (2) for $-\pi<\phi+\alpha<\pi$, $-\pi<\phi<\pi$; $Y(\phi+\alpha)-Y(\phi)$ is not continuous; and it is not at all clear if one can place (3) in a general form, i.e. in a form that does not depend on the particular representation

$$\hat{L}\to -\mathrm{i}\partial_\phi, \qquad \hat{\phi}\to\phi.$$

Therefore, since we cannot make a change of representation analogous to the change from the Schrödinger representation to the momentum representation, there seems little point in working with (4) or (5).

Let us consider instead the relations (see Carruthers and Nieto 1968)

$$[\sin\hat{\phi},\hat{L}]=\mathrm{i}\cos\hat{\phi}, \qquad [\cos\hat{\phi},\hat{L}]=-\mathrm{i}\sin\hat{\phi}, \tag{6}$$

where

$$(\hat{L}\psi)\phi=-\mathrm{i}\psi'(\phi), \qquad (\sin\hat{\phi}\,\psi)\phi=\sin\phi\,\psi(\phi),$$

which are valid on $\{\psi\,|\,\psi, \psi'$ abs. cont. $\psi(\pi)=\psi(-\pi)\}$. Proceeding formally (i.e. ignoring the fact that $\hat{L}$ is unbounded and that (6) is not valid everywhere in $\mathscr{H}$),

$$\mathrm{i}[\hat{L},\sin\hat{\phi}]=\cos\hat{\phi}$$

$$\Rightarrow[(\sin\hat{\phi})^n,\hat{L}]$$

$$=\sum_{m=0}^{n}(\sin\hat{\phi})^m[\sin\hat{\phi},\hat{L}](\sin\hat{\phi})^{n-m-1}$$

$$=\mathrm{i}n\cos\hat{\phi}(\sin\hat{\phi})^{n-1}$$

$$\therefore[\mathrm{e}^{-\mathrm{i}\alpha\sin\hat{\phi}},\hat{L}]=\alpha\cos\hat{\phi}\,\mathrm{e}^{-\mathrm{i}\alpha\sin\hat{\phi}}$$

$$\therefore\mathrm{e}^{-\mathrm{i}\alpha\sin\hat{\phi}}\hat{L}\,\mathrm{e}^{\mathrm{i}\alpha\sin\hat{\phi}}-\hat{L}=\alpha\cos\hat{\phi}$$

$$\Rightarrow\mathrm{e}^{-\mathrm{i}\alpha\sin\hat{\phi}}\,\mathrm{e}^{\mathrm{i}\beta\hat{L}}\,\mathrm{e}^{\mathrm{i}\alpha\sin\hat{\phi}}=\mathrm{e}^{\mathrm{i}\beta(\hat{L}+\alpha\cos\hat{\phi})} \qquad \forall\alpha,\beta\in\mathbb{R}. \tag{7}$$

Quite recently Braunss (1976) has published a discussion of the non-canonical commutation relation

$$\mathrm{i}[\hat{A},\hat{B}]=\hat{C}, \tag{8}$$

where $\hat{A}$, $\hat{B}$ are self-adjoint operators and $\hat{C}$ is a bounded self-adjoint operator on $\mathscr{H}$. The motivation for his work was the hope that one might be able to remove some of the divergences in field theory by replacing (1) by (8). We note, however, that there are examples of relations of the form (8) within conventional quantum mechanics:

(i) $[\hat{C},\hat{N}]=\mathrm{i}\hat{S}$, where $\hat{C}$, $\hat{S}$ are the cosine and sine operators and $\hat{N}$ is the number operator (Carruthers and Nieto 1968);

(ii) equations (6);

(iii) $[\sigma_x,\sigma_y]=2\mathrm{i}\sigma_z$.

Now Braunss has shown for bounded $\hat{A}$, $\hat{B}$ that

$$\mathrm{e}^{-\mathrm{i}\beta\hat{B}}\,\mathrm{e}^{\mathrm{i}\alpha\hat{A}}\,\mathrm{e}^{\mathrm{i}\beta\hat{B}}=\mathrm{e}^{\mathrm{i}\alpha(\hat{A}+\hat{C}_{\hat{B}}(\beta))}, \tag{9}$$

where

$$\hat{C}_{\hat{B}}(\beta)=\int_0^\beta \mathrm{e}^{-\mathrm{i}\sigma\hat{B}}\hat{C}\,\mathrm{e}^{\mathrm{i}\sigma\hat{B}}\,\mathrm{d}\sigma.$$

If we assume the validity of (9) even for one of $\hat{A}$ and $\hat{B}$ unbounded and let

$$\hat{A} = \hat{L}, \qquad \hat{B} = \sin \hat{\phi}, \qquad \hat{C} = \cos \hat{\phi},$$

then we have

$$\hat{C}_{\hat{B}}(\beta) = \int_0^\beta \mathrm{e}^{-\mathrm{i}\sigma \sin \hat{\phi}} \cos \hat{\phi}\, \mathrm{e}^{\mathrm{i}\sigma \sin \hat{\phi}}\, \mathrm{d}\sigma$$
$$= \beta \cos \hat{\phi},$$

which gives us (7) once again.

We may deduce from Braunss's work that (7) reduces to (3) on the appropriate dense set.

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